Examen Final : corrigé
ELE4203 — Robotique

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École Polytechnique de Montréal

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Question 1 (4 points) Calculons q_1 et q_2 pour $x = 0.75m$ et $y = 0.75m$:

$$
\begin{aligned}
K &= -0.53 \\
q_{2d}(1) &= 2.1294 \text{ rad } (122^\circ) \\
q_{1d}(1) &= 0.2622 \text{ rad } (15^\circ) \\
\boldsymbol{q}_d(1) &= \begin{bmatrix} 0.2622 \\ 2.1294 \end{bmatrix} \text{ rad} \\
{}^0\boldsymbol{J}^{-1} &= \begin{bmatrix} -0.8628 & 0.8039 \\ -1.4151 & -1.4151 \end{bmatrix} \\
\dot{\boldsymbol{q}}_d(1) &= {}^0\boldsymbol{J}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ m/s} \\
&= \begin{bmatrix} -1.6667 \\ 0 \end{bmatrix} \text{ rad/s} \\
{}^0\dot{\boldsymbol{J}} &= \begin{bmatrix} -a_2 C_{12}(\dot{q}_1 + \dot{q}_2) - a_1 C_1 \dot{q}_1 & -a_2 C_{12}(\dot{q}_1 + \dot{q}_2) \\ -a_2 S_{12}(\dot{q}_1 + \dot{q}_2) - a_1 S_1 \dot{q}_1 & -a_2 S_{12}(\dot{q}_1 + \dot{q}_2) \end{bmatrix} \\
&= \begin{bmatrix} 1.0000 & -0.6097 \\ 1.0000 & 0.5681 \end{bmatrix} \\
\ddot{\boldsymbol{q}}_d(t) &= {}^0\boldsymbol{J}^{-1}(\boldsymbol{q}_d) \left\{ \begin{bmatrix} \ddot{x}_d(t) \\ \ddot{y}_d(t) \end{bmatrix} - {}^0\dot{\boldsymbol{J}}(\boldsymbol{q}_d)\dot{\boldsymbol{q}}_d(t) \right\} \\
\ddot{\boldsymbol{q}}_d(1) &= {}^0\boldsymbol{J}^{-1} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ m/s}^2 - {}^0\dot{\boldsymbol{J}}\dot{\boldsymbol{q}}_d(1) \right\} \\
&= \begin{bmatrix} 1.5684 \\ -4.7170 \end{bmatrix} \text{ rad/s}^2
\end{aligned}
$$

Question 2 (6 points)

Soit deux constantes k_1 et k_2, nous avons

$$
\begin{aligned}
x_1 &= 0 \\
y_1 &= 0 \\
z_1 &= k_1 \\
x_2 &= -\left[D_1 + l_2 \cos(q_2)\right] \sin(q_1) \\
y_2 &= \left[D_1 + l_2 \cos(q_2)\right] \cos(q_1) \\
z_2 &= k_2 + l_2 \sin(q_2) \\
\dot{x}_1 &= \dot{y}_1 = \dot{z}_1 = 0 \\
v_1^2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_2 &= -\left[D_1 + l_2\cos(q_2)\right]\cos(q_1)\dot{q}_1 + l_2\sin(q_2)\sin(q_1)\dot{q}_2 \\
\dot{y}_2 &= -\left[D_1 + l_2\cos(q_2)\right]\sin(q_1)\dot{q}_1 - l_2\sin(q_2)\cos(q_1)\dot{q}_2 \\
\dot{z}_2 &= l_2\cos(q_2)\dot{q}_2 \\
v_2^2 &= \left[D_1 + l_2\cos(q_2)\right]^2\dot{q}_1^2 + l_2^2\dot{q}_2^2 \\
T_1 &= \frac{1}{2}I_1\dot{q}_1^2 \\
T_2 &= \frac{1}{2}m_2\left\{\left[D_1 + l_2\cos(q_2)\right]^2\dot{q}_1^2 + l_2^2\dot{q}_2^2\right\} \\
U_1 &= m_1 g k_1 \\
U_2 &= m_2 g[k_2 + l_2\sin(q_2)] \\
L &= \frac{1}{2}I_1\dot{q}_1^2 + \frac{1}{2}m_2\left\{\left[D_1 + l_2\cos(q_2)\right]^2\dot{q}_1^2 + l_2^2\dot{q}_2^2\right\} \\
&\quad - m_1 g k_1 - m_2 g[k_2 + l_2\sin(q_2)]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{q}_1} &= \{I_1 + m_2\left[D_1 + l_2\cos(q_2)\right]^2\}\dot{q}_1 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) &= \{I_1 + m_2\left[D_1 + l_2\cos(q_2)\right]^2\}\ddot{q}_1 - 2m_2 l_2\left[D_1 + l_2\cos(q_2)\right]\sin(q_2)\dot{q}_1\dot{q}_2 \\
\frac{\partial L}{\partial q_1} &= 0 \\
\tau_1 &= \{I_1 + m_2\left[D_1 + l_2\cos(q_2)\right]^2\}\ddot{q}_1 - 2m_2 l_2\left[D_1 + l_2\cos(q_2)\right]\sin(q_2)\dot{q}_1\dot{q}_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{q}_2} &= m_2 l_2^2\dot{q}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) &= m_2 l_2^2\ddot{q}_2 \\
\frac{\partial L}{\partial q_2} &= -m_2 l_2\left[D_1 + l_2\cos(q_2)\right]\dot{q}_1^2\sin(q_2) - m_2 g l_2\cos(q_2) \\
\tau_2 &= m_2 l_2^2\ddot{q}_2 + m_2 l_2\left[D_1 + l_2\cos(q_2)\right]\dot{q}_1^2\sin(q_2) + m_2 g l_2\cos(q_2)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} I_1 + m_2\left[D_1 + l_2\cos(q_2)\right]^2 & 0 \\ 0 & m_2 l_2^2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} -2m_2 l_2\left[D_1 + l_2\cos(q_2)\right]\sin(q_2)\dot{q}_1\dot{q}_2 \\ m_2 l_2\left[D_1 + l_2\cos(q_2)\right]\dot{q}_1^2\sin(q_2) \end{bmatrix} + g\begin{bmatrix} 0 \\ m_2 l_2\cos(q_2) \end{bmatrix}
\end{aligned}
$$

Question 3 (1 point)

Commande par couple précalculée.

Question 4 (1 point)

Les splines nous donnent des trajectoires continues en position, vitesse et accélération passant par plusieurs points.

Question 5 (2 points)

$t = 3/2$ et $h_i = 1$:

$$S_1(t) = a_1 + b_1(t - t_1) + c_1(t - t_1)^2 + d_1(t - t_1)^3$$

En X :

$$\begin{aligned}
a_1 &= 0 \\
c_1 &= 0 \\
b_1 &= (a_2 - a_1) - \frac{1}{3}(2c_1 + c_2) = -2 - \frac{1}{3}(0 + 3) = -3 \\
d_1 &= \frac{c_2 - c_1}{3} = 1 \\
x &= -\frac{3}{2} + \frac{1}{8} = -\frac{11}{8} \\
\frac{dx(t)}{dt} &= b_1 + 2c_1(t - t_1) + 3d_1(t - t_1)^2 = -3 + 0 + \frac{3}{4} = -\frac{9}{4}
\end{aligned}$$

En Y :

$$\begin{aligned}
a_1 &= 1 \\
c_1 &= -\frac{3}{2} \\
b_1 &= (a_2 - a_1) - \frac{1}{3}(2c_1 + c_2) = -1 - \frac{1}{3}(-3 + 0) = 0 \\
d_1 &= \frac{c_2 - c_1}{3} = \frac{1}{2} \\
y &= 1 - \frac{3}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{8} = \frac{11}{16} \\
\frac{dy(t)}{dt} &= b_1 + 2c_1(t - t_1) + 3d_1(t - t_1)^2 = 0 - \frac{3}{2} + \frac{3}{8} = -\frac{9}{8}
\end{aligned}$$

Vitesse tangentielle = 2.5156.

Question 6 (4 points)

a) Après avoir déterminé les centroïdes des balises en coordonnées pixel (après seuillage, étiquetage ou autre technique), il faut déterminer les éléments de la matrice :

$$\boldsymbol{P}\ {}^{C}\boldsymbol{T}_U \;=\; \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$$

par moindres carrés (avec $n \geq 6$ balises) :

$$\begin{bmatrix} x_{b1} & y_{b1} & z_0 & 1 & 0 & 0 & 0 & 0 & -i_{b1}x_{b1} & -i_{b1}y_{b1} & -i_{b1}z_0 \\ 0 & 0 & 0 & 0 & x_{b1} & y_{b1} & z_0 & 1 & -j_{b1}x_{b1} & -j_{b1}y_{b1} & -j_{b1}z_0 \\ x_{b2} & y_{b2} & z_0 & 1 & 0 & 0 & 0 & 0 & -i_{b2}x_{b2} & -i_{b2}y_{b2} & -i_{b2}z_0 \\ 0 & 0 & 0 & 0 & x_{b2} & y_{b2} & z_0 & 1 & -j_{b2}x_{b2} & -j_{b2}y_{b2} & -j_{b2}z_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{bn} & y_{bn} & z_0 & 1 & 0 & 0 & 0 & 0 & -i_{bn}x_{bn} & -i_{bn}y_{bn} & -i_{bn}z_0 \\ 0 & 0 & 0 & 0 & x_{bn} & y_{bn} & z_0 & 1 & -j_{bn}x_{bn} & -j_{bn}y_{bn} & -j_{bn}z_0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{41} \\ a_{42} \\ a_{43} \end{bmatrix} = \begin{bmatrix} i_{b1} \\ j_{b1} \\ i_{b2} \\ j_{b2} \\ \vdots \\ \vdots \\ i_{bn} \\ j_{bn} \end{bmatrix}$$

b) comme on connaît le $Z = z_0$ des points des objets, on a

$$\begin{bmatrix} ki_o \\ kj_o \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} ki_o \\ kj_o \\ k \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_0 \\ 1 \end{bmatrix}$$

Les inconnues sont x_o, y_o et k. On a donc

$$k \begin{bmatrix} i_o \\ j_o \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \\ a_{43} & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ 1 \end{bmatrix}$$

En isolant x_o, y_o et k, on obtient

$$\begin{bmatrix} -a_{11} & -a_{12} & i_o \\ -a_{21} & -a_{22} & j_o \\ -a_{41} & -a_{42} & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ k \end{bmatrix} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \\ a_{43} & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_o \\ y_o \\ k \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & i_o \\ -a_{21} & -a_{22} & j_o \\ -a_{41} & -a_{42} & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \\ a_{43} & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ 1 \end{bmatrix}$$

Question 7 (2 points)

Après calculs, on obtient

$$\begin{aligned}
m_{00} &= 25 \\
m_{01} = m_{10} &= 1 + 3 \cdot 2 + 5 \cdot 3 + 7 \cdot 4 + 5 \cdot 5 + 3 \cdot 6 + 1 \cdot 7 = 100 \\
\bar{i} &= 4 \\
\bar{j} &= 4 \\
\mu_{11} &= 0 \\
\mu_{02} = \mu_{20} &= 2(5 + 3(2^2) + 1(3^2)) = 52 \\
\phi_1 &= \frac{\mu_{20} + \mu_{02}}{(m_{00})^2} = 0.1664 \\
\phi_2 &= 0 \\
a &= b \\
\phi_{1\text{théor.}} &= \frac{1}{12}(1 + 1) = 0.1667 \\
\phi_{2\text{théor.}} &= 0
\end{aligned}$$

Dans ce cas, la discrétisation n'a pas influencé les valeurs.